

ON THE RADIUS AND THE ATTACHMENT NUMBER OF TETRAVALENT HALF-ARC-TRANSITIVE GRAPHS

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ABSTRACT. In this paper, we study the relationship between the radius r and the attachment number a of a tetravalent graph admitting a half-arc-transitive group of automorphisms. These two parameters were first introduced in [*J. Combin. Theory Ser. B* 73 (1998), 41–76], where among other things it was proved that a always divides $2r$. Intrigued by the empirical data from the census [*Ars Math. Contemp.* 8 (2015)] of all such graphs of order up to 1000 we pose the question of whether all examples for which a does not divide r are arc-transitive. We prove that the answer to this question is positive in the case when a is twice an odd number. In addition, we completely characterize the tetravalent graphs admitting a half-arc-transitive group with $r = 3$ and $a = 2$, and prove that they arise as non-sectional split 2-fold covers of line graphs of 2-arc-transitive cubic graphs.

1. INTRODUCTION

This paper stems from our research of finite simple connected tetravalent graphs that admit a group of automorphisms acting transitively on vertices and edges but not on the arcs of the graph; such groups of automorphisms are said to be *half-arc-transitive*. Observe that the full automorphism group $\text{Aut}(\Gamma)$ of such a graph Γ is then either arc-transitive or itself half-arc-transitive. In the latter case the graph Γ is called *half-arc-transitive*.

Tetravalent graphs admitting a half-arc-transitive group of automorphisms are surprisingly rich combinatorial objects with connections to several other areas of mathematics (see, for example, [1, 9, 10, 11, 13, 16, 18]). One of the most fruitful tools for analysing the structure of a tetravalent graph Γ admitting a half-arc-transitive group G is to study a certain G -invariant decomposition of the edge set $E(\Gamma)$ of Γ into the G -alternating cycles of some even length $2r$; the parameter r is then called the G -radius and denoted $\text{rad}_G(\Gamma)$ (see Section 2 for more detailed definitions). Since G is edge-transitive and the decomposition into G -alternating cycles is G -invariant, any two intersecting G -alternating cycles meet in the same number of vertices; this number is then called the *attachment number* and denoted $\text{att}_G(\Gamma)$. When $G = \text{Aut}(\Gamma)$ the subscript G will be omitted in the above notation.

It is well known and easy to see that $\text{att}_G(\Gamma)$ divides $2\text{rad}_G(\Gamma)$. However, for all known tetravalent half-arc-transitive graphs the attachment number in fact divides the radius. This brings us to the following question that we would like to propose and address in this paper:

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Question 1. *Is it true that the attachment number $\text{att}(\Gamma)$ of an arbitrary tetravalent half-arc-transitive graph Γ divides the radius $\text{rad}(\Gamma)$?*

By checking the complete list of all tetravalent half-arc-transitive graphs on up to 1000 vertices (see [15]), we see that the answer to the above question is affirmative for the graphs in that range. Further, as was proved in [14, Theorem 1.2], the question has an affirmative answer in the case $\text{att}(\Gamma) = 2$. In Section 3, we generalise this result by proving the following theorem.

Theorem 2. *Let Γ be a tetravalent half-arc-transitive graph. If its radius $\text{rad}(\Gamma)$ is odd, then $\text{att}(\Gamma)$ divides $\text{rad}(\Gamma)$. Consequently, if $\text{att}(\Gamma)$ is not divisible by 4, then $\text{att}(\Gamma)$ divides $\text{rad}(\Gamma)$.*

As a consequence of our second main result (Theorem 3) we see that, in contrast to Theorem 2, there exist infinitely many arc-transitive tetravalent graphs Γ admitting a half-arc-transitive group G with $\text{rad}_G(\Gamma) = 3$ and $\text{att}_G(\Gamma) = 2$. In fact, in Section 2, we characterise these graphs completely and prove the following theorem (see Section 2.2 for the definition of the dart graph).

Theorem 3. *Let Γ be a connected tetravalent graph. Then Γ is G -half-arc-transitive for some $G \leq \text{Aut}(\Gamma)$ with $\text{rad}_G(\Gamma) = 3$ and $\text{att}_G(\Gamma) = 2$ if and only if Γ is the dart graph of some 2-arc-transitive cubic graph.*

The third main result of this paper, stemming from our analysis of the situation described by Theorem 3, reveals a surprising connection to the theory of covering projections of graphs. This theory has become one of the central tools in the study of symmetries of graphs. A particularly thrilling development started with the seminal work of Malnič, Nedela and Škovič [5] who analysed the condition under which a given automorphism group of the base graph lifts along the covering projection. Recently, the question of determining the structure of the lifted group received a lot of attention (see [2, 6, 7]).

To be more precise, let $\varphi: \tilde{\Gamma} \rightarrow \Gamma$ be a covering projection of connected graphs and let $\text{CT}(\varphi)$ be the corresponding group of covering transformations (see [5], for example, for the definitions pertaining to the theory of graph covers). Furthermore, let $G \leq \text{Aut}(\Gamma)$ be a subgroup that lifts along φ . Then the lifted group \tilde{G} is an extension of $\text{CT}(\varphi)$ by G . If this extension is split then the covering projection φ is called G -split. The most natural way in which this can occur is that there exists a complement \tilde{G} of $\text{CT}(\varphi)$ in \tilde{G} and a \tilde{G} -invariant subset S of $V(\tilde{\Gamma})$, that intersects each fibre of φ in exactly one vertex. In such a case we say that S is a *section* for \tilde{G} and that \tilde{G} is a *sectional* complement of $\text{CT}(\varphi)$. Split covering projections without any sectional complement are called *non-sectional*. These turn out to be rather elusive and hard to analyse. To the best of our knowledge, the only known infinite family of non-sectional split covers was presented in [2, Section 4]. This family of non-sectional split covers involves cubic arc-transitive graphs of extremely large order.

In this paper we show that each connected tetravalent graph Γ admitting a half-arc-transitive group G of automorphisms such that $\text{att}_G(\Gamma) = 2$ and $\text{rad}_G(\Gamma) = 3$ is a 2-fold cover of the line graph of a cubic 2-arc-transitive graph, and that in the case when Γ is not bipartite the corresponding covering projection is non-sectional. This thus provides a new and rather simple infinite family of the somewhat mysterious case of non-sectional split covering projections (see Section 4 for more details).

2. HALF-ARC-TRANSITIVE GROUP ACTIONS ON GRAPHS

In the next two paragraphs we briefly review some concepts and results pertaining half-arc-transitive group actions on tetravalent graphs that we shall need in the remainder of this section. For more details see [8], where most of these notions were introduced.

A tetravalent graph Γ admitting a *half-arc-transitive* (that is vertex- and edge- but not arc-transitive) group of automorphisms G is said to be *G -half-arc-transitive*. The action of G induces two paired orientations of the edges of Γ and for any one of them each vertex of Γ is the head of two and the tail of the other two of its incident edges. (The fact that the edge uv is oriented from u to v will be denoted by $u \rightarrow v$.) A cycle of Γ for which every two consecutive edges either have a common head or common tail with respect to this orientation is called a *G -alternating cycle*. Since the action of G is vertex- and edge-transitive all of the G -alternating cycles have the same even length $2\text{rad}_G(\Gamma)$ and any two non-disjoint G -alternating cycles intersect in the same number $\text{att}_G(\Gamma)$ of vertices. These intersections, called the *G -attachment sets*, form an imprimitivity block system for the group G . The numbers $\text{rad}_G(\Gamma)$ and $\text{att}_G(\Gamma)$ are called the *G -radius* and *G -attachment number* of Γ , respectively. If $G = \text{Aut}(\Gamma)$ we suppress the prefix and subscript $\text{Aut}(\Gamma)$ in all of the above definitions.

It was shown in [8, Proposition 2.4] that a tetravalent G -half-arc-transitive graph Γ has at least three G -alternating cycles unless $\text{att}_G(\Gamma) = 2\text{rad}_G(\Gamma)$ in which case Γ is isomorphic to a particular Cayley graph of a cyclic group (and is thus arc-transitive). Moreover, in the case that Γ has at least three G -alternating cycles, $\text{att}_G(\Gamma) \leq \text{rad}_G(\Gamma)$ holds and $\text{att}_G(\Gamma)$ divides $2\text{rad}_G(\Gamma)$. In addition, the restriction of the action of G to any G -alternating cycle is isomorphic to the dihedral group of order $2\text{rad}_G(\Gamma)$ (or to the Klein 4-group in the case of $\text{rad}_G(\Gamma) = 2$) with the cyclic subgroup of order $\text{rad}_G(\Gamma)$ being the subgroup generated by a two-step rotation of the G -alternating cycle in question. In addition, if $C = (v_0, v_1, \dots, v_{2r-1})$ is a G -alternating cycle of Γ with $r = \text{rad}_G(\Gamma)$ and C' is the other G -alternating cycle of Γ containing v_0 then $C \cap C' = \{v_{i\ell} : 0 \leq i < a\}$ where $a = \text{att}_G(\Gamma)$ and $\ell = 2r/a$ (see [8, Proposition 2.6] and [12, Proposition 3.4]).

As mentioned in the Introduction one of the goals of this paper is to characterize the tetravalent G -half-arc-transitive graphs Γ with $\text{rad}_G(\Gamma) = 3$ and $\text{att}_G(\Gamma) = 2$. The bijective correspondence between such graphs and 2-arc-transitive cubic graphs (see Theorem 3) is given via two pairwise inverse constructions: the *graph of alternating cycles* construction and the *dart graph* construction. We first define the former.

2.1. The graph of alternating cycles. Let Γ be a tetravalent G -half-arc-transitive graph for some $G \leq \text{Aut}(\Gamma)$. The *graph of G -alternating cycles* $\text{Alt}_G(\Gamma)$ is the graph whose vertex set consists of all G -alternating cycles of Γ with two of them being adjacent whenever they have at least one vertex in common. We record some basic properties of the graph $\text{Alt}_G(\Gamma)$.

Proposition 4. *Let Γ be a connected tetravalent G -half-arc-transitive graph for some $G \leq \text{Aut}(\Gamma)$ having at least three G -alternating cycles. Then the graph $\text{Alt}_G(\Gamma)$ is a regular graph of valence $2\text{rad}_G(\Gamma)/\text{att}_G(\Gamma)$ and the induced action of G on $\text{Alt}_G(\Gamma)$ is vertex- and edge-transitive. Moreover, this action is arc-transitive if and only if $\text{rad}_G(\Gamma)$ does not divide $\text{att}_G(\Gamma)$.*

Proof. To simplify notation, denote $r = \text{rad}_G(\Gamma)$ and $a = \text{att}_G(\Gamma)$. Since each vertex of Γ lies on exactly two G -alternating cycles and the intersection of any two non-disjoint G -alternating cycles is of size a it is clear that each G -alternating cycle is adjacent to $\ell = 2r/a$ other G -alternating cycles in $\text{Alt}_G(\Gamma)$. Moreover, since G acts edge-transitively on Γ and each edge of Γ is contained in a unique G -alternating cycle, the induced action of G on $\text{Alt}_G(\Gamma)$ is vertex-transitive. That this action is also edge-transitive follows from the fact that G acts vertex-transitively on Γ and that the edges of $\text{Alt}_G(\Gamma)$ correspond to G -attachment sets of Γ .

For the rest of the proof fix one of the two paired orientations of Γ given by the action of G , let $C = (v_0, v_1, \dots, v_{2r-1})$ be a G -alternating cycle such that $v_0 \rightarrow v_1$ and let C' be the other G -alternating cycle containing v_0 , so that $C \cap C' = \{v_{i\ell} : 0 \leq i < a\}$. Since every other vertex of C is the tail of the two edges of C incident to it, the vertex v_ℓ is the tail of the two edges of C incident to it if and only if ℓ is even (in which case each $v_{i\ell}$ has this property).

Now, if ℓ is odd, then each element of G , mapping v_0 to v_ℓ necessarily interchanges C and C' , proving that in this case the induced action of G on $\text{Alt}_G(\Gamma)$ is in fact arc-transitive. We remark that this also follows from the fact, first observed by Tutte [19], that a vertex- and edge-transitive group of automorphisms of a graph of odd valence is necessarily arc-transitive. To complete the proof we thus only need to show that the induced action of G on $\text{Alt}_G(\Gamma)$ is not arc-transitive when ℓ is even. Recall that in this case each vertex $v_{i\ell} \in C \cap C'$ is the tail of the two edges of C incident to it. Therefore, since any element of G , mapping the pair $\{C, C'\}$ to itself of course preserves the intersection $C \cap C'$ it is clear that any such element fixes each of C and C' setwise, and so no element of G can interchange C and C' . This proves that the induced action of G on $\text{Alt}_G(\Gamma)$ is half-arc-transitive. \square

2.2. The dart graph and its relation to $\text{Alt}_G(\Gamma)$. The dart graph of a cubic graph was investigated in [4] (we remark that this construction can also be viewed as a special kind of the *arc graph* construction from [3]). Of course the dart graph construction can be applied to arbitrary graphs but here, as in [4], we are only interested in dart graphs of cubic graphs. We first recall the definition. Let Λ be a cubic graph. Then its *dart graph* $\text{Dart}(\Lambda)$ is the graph whose vertex set consists of all the arcs (called darts in [4]) of Λ with (u, v) adjacent to (u', v') if and only if either $u' = v$ but $u \neq v'$, or $u = v'$ but $u' \neq v$. In other words, the edges of $\text{Dart}(\Lambda)$ correspond to the 2-arcs of Λ . Note that this enables a natural orientation of the edges of $\text{Dart}(\Lambda)$ where the edge $(u, v)(v, w)$ is oriented from (u, v) to (v, w) .

Clearly, $\text{Aut}(\Lambda)$ can be viewed as a subgroup of $\text{Aut}(\text{Dart}(\Lambda))$ preserving the natural orientation. Furthermore, the permutation τ of $V(\text{Dart}(\Lambda))$, exchanging each (u, v) with (v, u) , is an orientation reversing automorphism of $\text{Dart}(\Lambda)$.

We now establish the correspondence between the 2-arc-transitive cubic graphs and the tetravalent graphs admitting a half-arc-transitive group of automorphisms with the corresponding radius 3 and attachment number 2. We do this in two steps.

Proposition 5. *Let Λ be a connected cubic graph admitting a 2-arc-transitive group of automorphisms G and let $\Gamma = \text{Dart}(\Lambda)$. Then Γ is a tetravalent G -half-arc-transitive graph such that $\text{rad}_G(\Gamma) = 3$ and $\text{att}_G(\Gamma) = 2$ with $\text{Alt}_G(\Gamma) \cong \Lambda$. Moreover, the natural orientation of Γ , viewed as $\text{Dart}(\Lambda)$, coincides with one of the two paired orientations induced by the action of G .*

Proof. That the natural action of G on Γ is half-arc-transitive can easily be verified (see also [4]). Now, fix an edge $(u, v)(v, w)$ of Γ and choose the G -induced orientation of Γ in such a way that $(u, v) \rightarrow (v, w)$. Since G is 2-arc-transitive on Λ , the other edge of Γ , for which (u, v) is its tail, is $(u, v)(v, w')$, where w' is the remaining neighbour of v in Λ (other than u and w). It is now clear that for each pair of adjacent vertices (x, y) and (y, z) of Γ the corresponding edge is oriented from (x, y) to (y, z) , and so the chosen G -induced orientation of Γ is the natural orientation of $\text{Dart}(\Lambda)$.

Finally, let v be a vertex of Λ and let u, u', u'' be its three neighbours. The G -alternating cycle of Γ containing the edge $(u, v)(v, u')$ is then clearly $C_v = ((u, v), (v, u'), (u'', v), (v, u), (u', v), (v, u''))$, implying that $\text{rad}_G(\Gamma) = 3$. This also shows that the G -alternating cycles of Γ naturally correspond to vertices of Λ . Since the three G -alternating cycles of Γ that have a nonempty intersection with C_v are the ones corresponding to the vertices u, u' and u'' , this correspondence in fact shows that $\text{Alt}_G(\Gamma)$ and Λ are isomorphic and that $\text{att}_G(\Gamma) = 2$. \square

Proposition 6. *Let Γ be a connected tetravalent G -half-arc-transitive graph for some $G \leq \text{Aut}(\Gamma)$ with $\text{rad}_G(\Gamma) = 3$ and $\text{att}_G(\Gamma) = 2$, and let $\Lambda = \text{Alt}_G(\Gamma)$. Then the group G induces a 2-arc-transitive action on Λ and $\text{Dart}(\Lambda) \cong \Gamma$. In fact, an isomorphism $\Psi: \text{Dart}(\Lambda) \rightarrow \Gamma$ exists which maps the natural orientation of $\text{Dart}(\Lambda)$ to a G -induced orientation of Γ .*

Proof. By Proposition 4 the graph Λ is cubic and the induced action of G on it is arc-transitive. Since $\text{rad}_G(\Gamma) = 3$ and $\text{att}_G(\Gamma) = 2$ it is easy to see that Γ and $\text{Dart}(\Lambda)$ are of the same order. Furthermore, let $C = (v_0, v_1, \dots, v_5)$ be a G -alternating cycle of Γ and C', C'', C''' be the other G -alternating cycles of Γ containing v_0, v_1 and v_5 , respectively. Then $C \cap C' = \{v_0, v_3\}$, $C \cap C'' = \{v_1, v_4\}$ and $C \cap C''' = \{v_2, v_5\}$. It is thus clear that any element of G , fixing v_0 and mapping v_1 to v_5 (which exists since C is G -alternating and G is edge-transitive on Γ), fixes both C and C' but maps C'' to C''' . Therefore, the induced action of G on Λ is 2-arc-transitive.

To complete the proof we exhibit a particular isomorphism $\Psi: \text{Dart}(\Lambda) \rightarrow \Gamma$. Fix an orientation of the edges of Γ , induced by the action of G , and let C and C' be two G -alternating cycles of Γ with a nonempty intersection. Then (C, C') and (C', C) are vertices of $\text{Dart}(\Lambda)$. Let $C \cap C' = \{u, u'\}$ and observe that precisely one of u and u' is the head of both of the edges of C incident to it. Without loss of generality assume it is u . Then of course u' is the head of both of the edges of C' incident to it. We then set $\Psi((C, C')) = u$ and $\Psi((C', C)) = u'$. Therefore, for non-disjoint G -alternating cycles C and C' of Γ we map (C, C') to the unique vertex in $C \cap C'$ which is the head of both of the edges of C incident to it. Since each pair of non-disjoint G -alternating cycles meets in precisely two vertices and each vertex of Γ belongs to two G -alternating cycles of Γ , this mapping is injective and thus also bijective. We now only need to show that it preserves adjacency and maps the natural orientation of $\text{Dart}(\Lambda)$ to the chosen G -induced orientation of Γ . To this end let C, C' and C'' be three G -alternating cycles of Γ such that C has a nonempty intersection with both C' and C'' . Recall that then the edge $(C', C)(C, C'')$ is oriented from (C', C) to (C, C'') in the natural orientation of $\text{Dart}(\Lambda)$. Denote $C = (v_0, v_1, \dots, v_5)$ and without loss of generality assume $C \cap C' = \{v_0, v_3\}$ and $C \cap C'' = \{v_1, v_4\}$.

Suppose first that $v_0 \rightarrow v_1$. Then v_0 is the head of both of the edges of C' incident to it, and so $\Psi((C', C)) = v_0$. Similarly, v_1 is the head of both of the edges of C incident to it, and so $\Psi((C, C'')) = v_1$. If on the other hand $v_1 \rightarrow v_0$, then $\Psi((C', C)) = v_3$ and $\Psi((C, C'')) = v_4$. In both cases, Ψ maps the oriented edge $(C', C)(C, C'')$ to an oriented edge of Γ , proving that it is an isomorphism of graphs, mapping the the natural orientation of $\text{Dart}(\Lambda)$ to the chosen G -induced orientation of Γ . \square

Theorem 3 now follows directly from Propositions 5 and 6.

3. PARTIAL ANSWER TO QUESTION 1 AND PROOF OF THEOREM 2

In this section we prove Theorem 2 giving a partial answer to Question 1. We first prove an auxiliary result.

Proposition 7. *Let Γ be a tetravalent G -half-arc-transitive graph with $\text{att}_G(\Gamma)$ even. Then for each vertex v of Γ and the two G -alternating cycles C and C' , containing v , the antipodal vertex of v on C coincides with the antipodal vertex of v on C' . Moreover, the involution τ interchanging each pair of antipodal vertices on all G -alternating cycles of Γ is an automorphism of Γ centralising G .*

Proof. Denote $r = \text{rad}_G(\Gamma)$ and $a = \text{att}_G(\Gamma)$. Let v be a vertex of Γ and let C and C' be the two G -alternating cycles of Γ containing v . Denote $C = (v_0, v_1, \dots, v_{2r-1})$ with $v = v_0$. Recall that then $C \cap C' = \{v_{i\ell} : 0 \leq i < a\}$, where $\ell = 2r/a$. Since a is even $v_r \in C \cap C'$. Now, take any element $g \in G_v$ interchanging v_1 with v_{2r-1} as well as the other two neighbours of v (which are of course neighbours of v on C'). Then g reflects both C and C' with respect to v . Since v_r is antipodal to v on C , it must be fixed by g , but since v_r is also contained in C' , this implies that it is in fact also the antipodal vertex of v on C' . This shows that for each G -alternating cycle C and each vertex v of C the vertex v and its antipodal counterpart on C both belong to the same pair of G -alternating cycles (this implies that the G -transversals, as they were defined in [8], are of length 2) and are also antipodal on the other G -alternating cycle containing them.

It is now clear that τ is a well defined involution on the vertex set of Γ . Since the antipodal vertex of a neighbor v_1 of $v = v_0$ on C is the neighbor v_{r+1} of the antipodal vertex v_r , it is clear that τ is in fact an automorphism of Γ . Since any element of G maps G -alternating cycles to G -alternating cycles it is clear that τ centralises G . \square

We are now ready to prove Theorem 2. Let Γ be a tetravalent half-arc-transitive graph. Denote $r = \text{rad}(\Gamma)$ and $a = \text{att}(\Gamma)$, and assume r is odd. Recall that a divides $2r$. We thus only need to prove that a is odd. Suppose to the contrary that a is even, and so by assumption $a \equiv 2 \pmod{4}$. Then the graph Γ admits the automorphism τ from Proposition 7. Now, fix one of the two paired orientations of the edges induced by the action of $\text{Aut}(\Gamma)$ and let $C = (v_0, v_1, \dots, v_{2r-1})$ be an alternating cycle of Γ with v_0 being the tail of the edge v_0v_1 . Since $v_0^\tau = v_r$ and $v_1^\tau = v_{r+1}$ it follows that v_r is the tail of the edge v_rv_{r+1} . But since r is odd this contradicts the fact that every other vertex of C is the tail of the two edges of C incident to it. Thus a is odd, as claimed.

To prove the second part of the theorem assume that a is not divisible by 4. If r is even then the fact that a divides $2r$ implies that a divides r as well. If however r is odd, we can apply the first part of the theorem. This completes the proof.

4. AN INFINITE FAMILY OF NON-SECTIONAL SPLIT COVERS

As announced in the introduction, tetravalent G -half-arc-transitive graphs Γ with $\text{rad}_G(\Gamma) = 3$ and $\text{att}_G(\Gamma) = 2$ yield surprising examples of the elusive non-sectional split covers. In this section, we present this connection in some detail.

Theorem 8. *Let Γ be a connected non-bipartite G -half-arc-transitive graph Γ of order greater than 12 with $\text{rad}_G(\Gamma) = 3$ and $\text{att}_G(\Gamma) = 2$. Then there exists a 2-fold covering projection $\wp: \Gamma \rightarrow \Gamma'$ and an arc-transitive group $H \leq \text{Aut}(\Gamma')$ which lifts along \wp in such a way that Γ is a non-sectional H -split cover of Γ' .*

Proof. Since $\text{att}_G(\Gamma) = 2$, each G -attachment set consists of a pair of antipodal vertices on a G -alternating cycle of Γ . Let \mathcal{B} be the set of all G -attachment sets in Γ . By Proposition 7, there exists an automorphism τ of Γ centralising G , which interchanges the two vertices in each element of \mathcal{B} . Let $\tilde{G} = \langle G, \tau \rangle$ and note that \tilde{G} acts transitively on the arcs of Γ . Since τ is an involution centralising G not contained in G , we see that $\tilde{G} = G \times \langle \tau \rangle$.

Let Γ' be the quotient graph with respect to the group $\langle \tau \rangle$, that is, the graph whose vertices are the orbits of $\langle \tau \rangle$ and with two such orbits adjacent whenever they are joined by an edge in Γ . Since \tilde{G} is arc-transitive and $\langle \tau \rangle$ is normal in \tilde{G} , each $\langle \tau \rangle$ -orbit is an independent set. Moreover, if two $\langle \tau \rangle$ -orbits B and C are adjacent in Γ' , then the induced subgraph $\Gamma[B \cup C]$ is clearly vertex- and arc-transitive and is thus either $K_{2,2}$ or $2K_2$. In the former case, it is easy to see that Γ is isomorphic to the lexicographic product of a cycle with the edge-less graph on two vertices. Since $\text{rad}_G(\Gamma) = 3$ and the orbits of $\langle \tau \rangle$ coincide with the elements of \mathcal{B} , this implies that Γ has only 6 vertices, contradicting our assumption on the order of Γ . This contradiction implies that $\Gamma[B \cup C] \cong 2K_2$ for any pair of adjacent $\langle \tau \rangle$ -orbits B and C , and hence the quotient projection $\wp: \Gamma \rightarrow \Gamma'$ is a 2-fold covering projection with $\langle \tau \rangle$ being its group of covering transformations.

Since τ normalises G , the group \tilde{G} projects along \wp and the quotient group $H = \tilde{G}/\langle \tau \rangle$ acts faithfully as an arc-transitive group of automorphisms on Γ' . In particular, since the group of covering projection $\langle \tau \rangle$ has a complement G in \tilde{G} , the covering projection \wp is H -split.

By [2, Proposition 3.3], if \wp had a sectional complement with respect to H , then Γ would be a canonical double cover of Γ' , contradicting the assumption that Γ is not bipartite. \square

REMARK. In [4, Proposition 9] it was shown that a cubic graph Λ is bipartite if and only if $\text{Dart}(\Lambda)$ is bipartite. Since there exist infinitely many connected non-bipartite cubic 2-arc-transitive graphs, Theorem 3 thus implies that there are indeed infinitely many connected non-bipartite G -half-arc-transitive graphs Γ with $\text{rad}_G(\Gamma) = 3$ and $\text{att}_G(\Gamma) = 2$. In view of Theorem 8, these yield infinitely many non-sectional split covers, as announced in the introduction. Furthermore, note that the G -alternating 6-cycles in the graph Γ appearing in the proof of the above theorem project by \wp to cycles of length 3, implying that Γ' is a tetravalent arc-transitive graph of girth 3. Since it is assumed that the order of Γ is larger than 12 (and thus the order of Γ' is larger than 6), we may now use [17, Theorem 5.1] to conclude that Γ' is isomorphic to the line graph of a 2-arc-transitive cubic graph.

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